in the neighbourhood of stable rotations.

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ON THE NON-UNIQUENESS. OF NON-LINEAR WAVE SOLUTIONS IN A VISCOUS LAYER*

A.V. BUNOV, E.A. DEMEKHIN, and V.YA. SHKADOV

Solutions of the stationary travelling wave type are considered in draining layers of a viscous fluid. A one-parameter family of waves /1/ is studied that softly branches off into the upper branch of the neutral stability curve of the plane-parallel flow and goes over into a negative soliton (phase velocity $c<3$) as the wave number tends to zero. It is shown that this family is not unique: for small values of the parameter δ characterizing the mass flow rate, a second and third family of waves branches off from it with half the period. The critical value $\delta=\delta_*$ is found for which the bifurcation points of the second and third families merge, while for $\delta>\delta_*$ they go into the complex plane; a dependence of the wave number on δ for which the bifurcation occurs is obtained analytically. The properties of the second family of the periodic wave and positive soliton type, for which $c>3$ are studied. The solutions are constructed numerically: the periodic solutions are continued in the parameter from the bifurcation points or from the known solutions by using the method of invariant imbedding; the soliton solutions are constructed by joining the linear asymptotic forms as the values of the longitudinal coordinate tend to infinity.

1. Steady wave motions of a viscous fluid in a plane layer on a vertical surface are described in the long-wave approximation by the equation /2, 3/

$$h^3h''' + \delta\,[6\,(q-c)^2 - c^2h^2]\,h' + [h^3 - q - c\,(h-1)] = 0 \tag{1.1}$$
$$\delta = 3^{-7/9}5^{-1}\gamma^{-1/9}R^{11/9}, \quad \gamma = \sigma\rho^{-1}\nu^{-4/3}g^{-1/3}$$

Here $h(x)$ is the layer thickness, q is the mean flow rate, c is the phase velocity referred to the mean flow rate velocity of the laminar waveless flow, σ is the surface tension, R is Reynolds number calculated from the mean flow rate and the layer thickness corresponding to waveless flow, and x is the longitudinal coordinate.

The conditions for periodic waves

$$h(0) = h\left(\frac{2\pi}{\alpha}\right), \quad h'(0) = h'\left(\frac{2\pi}{\alpha}\right), \quad h''(0) = h''\left(\frac{2\pi}{\alpha}\right), \quad \frac{\alpha}{2\pi}\int_0^{2\pi/\alpha} h\,dx = 1 \tag{1.2}$$

and for solitary waves (solitons)

$$h\to 1, \; h^{(n)}\to 0 \;\text{ as }\; x\to\pm\infty \tag{1.3}$$

The trivial solution $h(x)\equiv 1,\ q=1$ corresponds to a plane-parallel waveless flow. As is shown in /2/, a selfoscillating wave solution branches off softly from the trivial solution at the point $\alpha_0=\sqrt{15\delta}$. The fundamental properties of these solutions are investigated in /2, 4/.

Introducing the small parameter ε, we obtain the following expansion in the semicircle $\alpha=\alpha_0$

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$$s = \alpha/\alpha_0 = 1 - \varepsilon^2, \; c = 3 - 12{,}3\varepsilon^2\beta^2, \; q = 1 + 6\varepsilon^2\beta^2$$
$$\beta^2 = \frac{2}{3}\alpha_0^6(1 + 4{,}14\alpha_0^6)^{-1}, \quad h = 1 + 2\varepsilon\beta\sin x - \frac{\varepsilon^2\beta^2}{\alpha_0^3}\sin 2x + \frac{7}{5}\varepsilon^2\beta^2\cos 2x \tag{1.4}$$

As $\varepsilon \to 0$ the expansion (1.4) tends to the exact solution. Formulas (1.4) can also be obtained from the solution in /2/ by an expansion in the amplitude ε.

To continue the solution (1.4) in the parameter $s = \alpha/\alpha_0$, we use the method of invariant imbedding /5/. We introduce compression of the independent variable $x \to \alpha x$, so that $x \in [0, 2\pi]$. The periodic solution is sought in the form

$$h = \sum_{k=-N}^{N} h_k e^{ikx}, \quad h_{-k} = \bar{h}_k, \quad h_0 = 1, \quad \mathrm{Im}\{h_1\} = 0 \tag{1.5}$$

Substituting (1.5) into (1.1) and collecting terms for identical harmonics, we obtain a non-linear system of $2N+1$ equations in $2N+1$ unknowns q, c, h_k $(k = 1, 2, \ldots, N)$. The quantities α and δ are considered given. The number of harmonics was determined by the condition $N = 4$ antier (α/α_0); here $|h_N/h_1| < 10^{-3}$ in the majority of cases.

We supplement the system by giving the curve $\alpha = \alpha(\lambda), \delta = \delta(\lambda)$ in the parameter space, where λ is a certain parameter. Differentiating the non-linear system with respect to λ, we obtain a system of $2N+1$ ordinary differential equations. If the solution is given at a certain point of the curve, it can be continued by numerical integration on the whole curve $\alpha = \alpha(\lambda), \delta = \delta(\lambda)$ up to a singularity, the bifurcation point, say. We note that the deformation of the curve can successfully reduce system degeneration in a number of cases, and the singularity can be bypassed. The method mentioned enabled us to find the solution in domains with abrupt changes in the desired functions from the parameters of the problem.

The first family of wave solutions in the domain $s \in (s_0, 1), \delta \in (0, 1)$ was studied in detail by this method with the initial data (1.4). For $\delta > 1$ the solution does not actually differ from the asymptotic form as $\delta \to \infty$. The periodic solution is successfully continued to $s \approx 0.1$; the periodic wave differs slightly from a solitary wave for this value of s. For $s = 0$ the family is supplemented by a negative soliton. Since the phase velocity of a soliton referred to the first family is $c < 3$, then we agree to call it negative unlike the positive soliton for which $c > 3$.

We will use the method in /4/ to find the soliton solution. Since $h_1 = h - 1 \to 0$ as $x \to \pm\infty$, after linearization (1.1) becomes

$$h_1''' + \rho\omega h_1' + (3 - c)h_1 = 0, \quad \omega = 5c^2 - 12c + 6 \tag{1.6}$$

which has three linearly-independent solutions. It can be shown from an analysis of the characteristic polynomial (1.6) that for $\delta < 2.38$ one of the soliton fronts corresponding to the two complex-conjugate roots of the polynomial will oscillate: forward for $c > 3$ and backward for $c < 3$.

For fixed δ let the approximate value $c < 3$ be known. The roots of the characteristic polynomial are $\sigma_1 = 2m < 0$, $\sigma_{2,3} = -m \pm i\beta$. By virtue of the damping of the solutions as $x \to +\infty$, we can take $h = 1 + \varepsilon$, $h' = 2\varepsilon m$, $h'' = 4\varepsilon m^2$, $\varepsilon = 0.01$ as initial conditions. Integrating (1.1) from these initial conditions to smaller x, we arrive at a domain where the asymptotic form $x \to -\infty$ is valid (c is close to the eigenvalue), and therefore the solution has the form

$$h \approx 1 + Ae^{2mx} + B_e^{-mx}\sin(\beta x + \psi),$$

$(A, B, \psi$ can be expressed in terms of $h, h', h'')$. If c is the eigenvalue, then the solution that grows as $x \to -\infty$ is suppressed, i.e., $A(c)$ vanishes. It can be shown that for A to vanish it is necessary that $(m^2 + \beta^2)(h - 1) + 2mh' + h'' = 0$. During the computations we select c such that this condition will be satisfied.

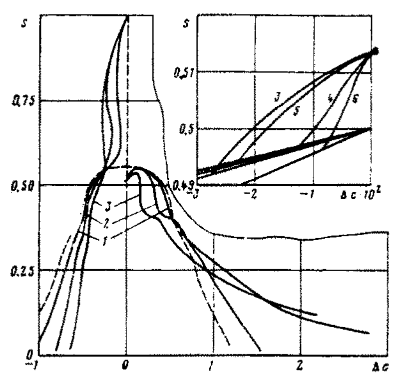

Fig.1

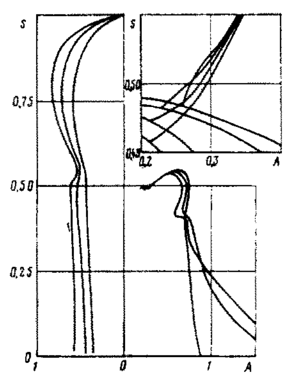

Fig.2

We proceed in an analogous manner for the positive soliton $c > 3$ but we perform the numerical integration of (1.1) towards larger x. For $c = 3$ the solitary wave does not exist: integrating (1.1) with respect to x for $c = 3$ we obtain the relationship

$$\int_{-\infty}^{\infty} (h-1)^2(h+2)\,h^{-3}\,dx = 0$$

which is satisfied only for $h(x) = 1$.

To represent the results of specific computations for small δ, we introduce the independent variables Δc, Q, $H(X)$:

$$c = 3 + \alpha_0^3\Delta c, \quad q = 1 + \alpha_0^6 Q, \quad h = 1 + \alpha_0^3 H, \quad \frac{d}{dx} = \alpha_0 \frac{d}{dX}$$

The capital letters refer to the stretched variables, and the lower-case letters to those not stretched. Then as $\delta \to 0$, the (1.1) in stretched variables becomes the model equation /6, 7/

$$H''' + H' - \Delta c H + 3H^2 = Q \tag{1.7}$$

Curves of Δc, $A = H_+ - H_-$ against s and δ are shown in Figs.1 and 2 (on the left) for the first family. The numbers on the curves correspond to values of $\delta \cdot 10^2$. The dashed lines correspond to the value $\delta = 0$.

As δ increases the wave characteristics differ slightly for different δ. A definite distinction between the phase velocities, flow rates, and amplitudes, which also decreases as δ increases, is conserved only in the long-wave part. For $\delta = 1$ the solution at $s > 0.25$ is in good agreement with the asymptotic form $\delta = \infty$.

As is seen from Figs.1 and 2, for small δ there are two amplitude maxima: one larger in the optimal ($s \approx 0.8$) regime domain /2/, and the other smaller for $s \approx 0.4$. As δ increases the small maximum vanishes, while the larger starts to shift toward $s = 0$.

The evolution of the form of the first family is shown in Fig.3 for $\delta = 0.04$ as s diminishes. As $\delta \to \infty$, the wave shape becomes symmetric, and the oscillations characteristic for small s drop into a trough. For small s, large δ correspond to the build-up of the symmetric shape.

We give below the parameters of the negative solitons: the phase velocity c and the amplitude $a = h_+ - h_-$ (the plus and minus subscripts denote the greatest and least values of the quantities)

δ	0.01	0.03	0.06	0.1	0.2	0.4	1.0
c	2.93	2.73	2.47	2.22	1.91	1.65	1.45
a	0.036	0.149	0.285	0.446	0.649	0.847	0.971

These results agree with the results in /4/ for the values of the phase velocity.

2. Selfoscillating solutions of the first family with periods $\pi, 2\pi/3, \ldots, 2\pi/n, \ldots$, respectively branch off from the trivial solution from the points $s = 1/2, 1/3, \ldots, 1/n, \ldots$ In particular, the solution obtained from (1.4) by the substitution $x \to 2x$ emerges from $s = 1/2$

$$s = \frac{1}{2}(1 - \varepsilon^2), \quad h = 1 + 2\varepsilon\beta \sin 2x - \frac{\varepsilon^2\beta^2}{\alpha_0^3} \sin 4x + \frac{7}{5}\varepsilon^2\beta^2 \cos 4x \tag{2.1}$$

Let us consider the π-periodic solution (2.1) as a degenerating 2π-periodic solution with zero odd harmonics. In principle, a selfoscillatory solution with non-zero odd harmonics can branch off from the degenerating solution. We impose a perturbation $h \to h + \mu f$, $q \to q + \mu U$, $c \to c + \mu\delta c$ on the solution of the first family. After substitution into (1.1) and passage to the limit, we obtain the equation

$$Af''' + Bf' + Df + Ru + G\delta c = 0$$
$$A = \alpha^3 h^3, \quad B = \alpha\delta\,[6(q-c)^2 + c^2h^2]$$
$$D = 3\alpha^3 h^2 h''' - 2\alpha\delta c^2 hh' + 3h^2 - c$$
$$R = 12\alpha\delta(q-c)h' - 1$$
$$G = -2\alpha\delta h'\,[6(q-c) + c^2h^2] + 1 - h$$

The existence of 2π-periodic non-trivial solutions of the following linear problem with π-periodic coefficients

$$Af''' + Bf' + Df = 0$$
$$f(0) = f(2\pi), \quad f'(0) = f'(2\pi), \quad f''(0) = f''(2\pi) \tag{2.2}$$

is the bifurcation condition.

We assume that the branching occurs in a small but finite neighbourhood of $s = 1/2$ so that the solution (2.1) can be used. We seek f in the form

$$f = \sum_{k=1}^{4} (F_k \sin kx + \Phi_k \cos kx)$$

Substituting the expression for f into (2.2) and rewriting the coefficients A, B, D using (2.1), we obtain a system of linear eighth-order algebraic equations in F_k, Φ_k with zero right side. Therefore, the branching condition is the degeneracy of the principal eighth-order matrix which will not be written down because of its awkwardness. Furthermore, it can be shown that the even and odd F_k, Φ_k enter the equation independently, and it is sufficient to take a fourth-order matrix generated by the odd harmonics $\sin x, \cos x, \sin 3x, \cos 3x$ for the branching condition. After evaluation of the determinant of the matrix mentioned, and neglecting terms of order greater than ε^4, which is equivalent to conserving terms of order ε^2 in (2.1) we obtain the final form of the branching condition

$$a\varepsilon^4 + b\varepsilon^2 + 1 = 0 \tag{2.3}$$
$$a = 134{,}94\,(23{,}05 + 27{,}6\alpha_0^6 + \alpha_0^{12})(1 + 4{,}14\,\alpha_0^6)^{-2}$$
$$b = -20{,}79\,(12{,}74 + \alpha_0^6)(1 + 4{,}14\alpha_0^6)^{-1}, \quad \alpha_0 = \sqrt{15\delta}$$

The dependence $\varepsilon^2 = \varepsilon^2(\delta)$ is given below

$\delta\cdot 10^2$	0	2	4	6	8	10	12
$\varepsilon_1^2\cdot 10^3$	4,2	4,7	6,88	17	33	60	99
$\varepsilon_2^2\cdot 10^2$	7,6	8,6	11	16	20	22	22

Note that the assumption about the smallness of ε^2 introduced at the beginning of the discussion is satisfied. For $\delta > \delta_* \approx 0.138$ there are no real solutions of (2.3); therefore, for $\delta \in (0, \delta_*)$ two more branch off from the first family, while for $\delta > \delta_*$ these last families merge into one.

We later limit ourselves to the smaller root.

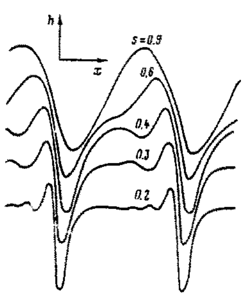

Fig.3

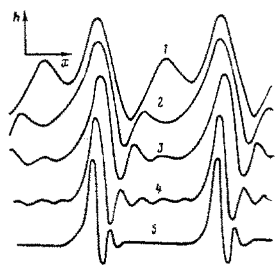

Fig.4

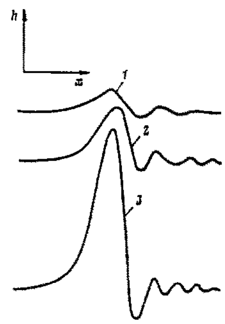

Fig.5

We agree to call the family that branches off here the second family. The neighbourhood of the bifurcation point is given in Figs.1 and 2. The first family emerges from the point $s = 1/2$ at twice the frequency, and the second branches off from this family. The dependence of $\Delta c, A = H_+ - H_-$ on s and δ is shown in Figs.1 and 2 (on the right) for the second family. As is seen from the graphs, the normalized wave number s first starts to increase as one moves away from the bifurcation point, and then diminishes after passing through the maximum ($s \in (0.53, 0.58)$ for the δ considered). As $s \to 0$ the second family can become, in particular, a positive soliton. The second family of waves was observed in experiment /8, 9/.

The evolution of the shape of the family is displayed in Fig.4 for removal from the bifurcation point and the gradual passage to a positive soliton for $\delta = 0.04$ *1* — $\Delta c = 1.5\cdot 10^{-3}$, *2* — 0.363, *3* — 0.711, *4* — 1.059, *5* — 1.41; here the dependence on the velocity $\Delta c = (c-3)/\alpha_0^3$ is taken because of the ambiguous dependence on s. In a small neighbourhood of the bifurcation point $c < 3$, mainly for the second family $c > 3$. For $\delta = 0$ (the dashed line in Fig.1) and $s > 0.58$ there is one branch $\Delta c = 0$; for $s < 0.58$ there are three branches emerging from the branch point: $\Delta c = 0, c > 3, c < 3$. (Two such branches are found in /10/ for this case.)

The branch $\Delta c = 0$ reaches the point $s = 0.4979$, which agrees with the value obtained by numerical continuation of the family from $s = 1$ in /11/. For $\delta = 0$ the bifurcation is not the bifurcation of a common position and it holds because of the high symmetry of (1.6); for small movements of the parameter δ the bifurcation dissociates (/12, p.120) and we then have two first and second family branches. As δ increases, the asymmetry of the families grows, it is most strongly apparent in the soliton solutions (see Fig.5, positive solitons: *1* — δ = 0.02, *2* — 0.03, *3* — 0.0392). For $\delta = 0.0392$ the amplitude of the positive soliton is five times greater, say, than the amplitude of the negative soliton. The dependence of the phase velocity and amplitude is presented below

δ	0.01	0.02	0.03	0.035	0.0392
c	3.076	3.26	3.65	4.09	4.96
a	0.041	0.136	0.336	0.563	0.976

The papers /10, 13/ are devoted to an investigation of soliton solutions for the model equation, and the paper /14/ is devoted to non-stationary solutions of solitary wave type.

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INFLUENCE OF NARROW CYLINDRICAL CAVITIES ON THE WAVE FIELD EXCITED BY A CONCENTRATED FORCE IN AN ELASTIC SPACE*

T.V. SUVOROVA

An elasticity theory problem is considered concerning the excitation of a wave field in a space weakened by a system of cylindircal cavities of small radius with rigid walls, with a concentrated force applied to a certain point of the space outside the shafts and varying sinusoidally. The solution of this problem is constructed by the principle of superposing the solutions of the following problems: the non-axisymmetric vibrations of an elastic space subjected to an oscillating concentrated force (problem 1); the wave field that occurs in an elastic space perforated by a system of narrow cavities vibrating under the effect of a sinusoidally varying stress applied to their walls (problem 2).

We also apply the method elucidated below to the investigation of the displacement field in an elastic space equipped with a system of elastic cylindrical inclusions of small diameter, or a system of cavities filled with liquid or a viscoelastic medium.

1. We consider problem 1. We obtain formulas describing the wave field in a space excited by a concentrated force $\mathbf{X}e^{-i\omega t}$ ($\mathbf{X} = \{X_1, X_2, X_3\}$, ω is the vibration frequency) applied to a

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